PROPERTIES OF ELECTRIC FIELDS IN INHOMOGENEOUS CONDUCTING MEDIA LOCATED IN A STRONG EXTERNAL MAGNETIC FIELD

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Investigations of the effective parameters and characteristics of electric fields in inhomogeneous media have long attracted a great deal of attention in connection with the study of semiconductor compounds and composite materials [1]. Analogous questions arise in inhomogeneous plasmas. Depending on the character of the inhomogeneities, the generalized properties of the medium can differ strongly from their local values; fluctuations of the electric fields may also be found to be considerable, particularly near the threshold of flow-through [2, 3].

In strong magnetic fields, where the ratio of the harmonic frequency of the rotation of the current carrier ω to the frequency of collisions ν is great ($\beta = \omega/\nu > 1$), the picture of the distribution of the current in an inhomogeneous medium is considerably more complicated. In this case, the conductivity becomes a tensor quantity and even very weak fluctuations of the properties of the medium lead to a strong perturbation of the local electric fields, as a result of which the effective parameters undergo a radical change [4-7].

The structure of heterogeneous media can be different. Along with media with localized inclusions and media with regular inhomogeneities, characteristic for a solid body, media with randomly inhomogeneous properties are frequently encountered. The latter are typical to a great extent for a low-temperature plasma, mechanical mixtures, and polycrystals.

Depending on the structure of the inhomogeneities, there are different theoretical methods of investigation. With a consideration of simple structures, the usual method consists in the use of the theory of boundary-value problems for calculation of the electric fields and their subsequent averaging [8, 9]. The solutions obtained by this method for isolated inclusions are extended to the case of media with a large concentration of inhomogenities using the method of self-consistency [10]. The validity of such an approach is based on a comparison with several exact results [5, 6]. For randomly inhomogeneous media, the natural apparatus for the calculation is the theory of random fluctuations, which is used in the work of a number of authors [7, 11-13].

In the present work the theory of random functions and the theory of boundary-value problems are used to investigate the characteristics of continuously inhomogeneous media in a magnetic field; the greatest amount of attention is paid to the little-studied question of the special characteristics of the formation of local electric fields in such media and their statistical characteristics.

1. Let us consider a conducting two-phase medium with a statistically homogeneous and isotropic distribution of small-scale regions of raised and lowered conductivity, which are oriented along the magnetic field. It is postulated that the dimension of the inhomogeneities is considerably greater than the length of the free-flight path.

The steady-state distribution of the current density $\mathbf{j} = \{j_x(x, y), j_y(x, y)\}$ and the intensity of the electric field $\mathbf{e} = \{e_x(x, y), e_y(x, y)\}$ in such a medium are described by the system of equations

$$\operatorname{div}\mathbf{j} = 0, \quad \operatorname{rot}\mathbf{e} = 0, \quad \hat{\sigma} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}, \quad \mathbf{j} = \hat{\sigma}\mathbf{e}, \tag{1.1}$$

where $p = \sigma/(1+\beta^2)$, $q = \sigma\beta/(1+\beta^2)$ are the components of the tensor of the conductivity, defined in terms of the conductivity σ in the absence of a magnetic field and the parameter β. Both parameters σ and β are random functions of the coordinates x and y.

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The general problem consists in determination of the tensor of the effective conductivity σ_e, which connects the fields averaged with respect to the volume

$$\langle \mathbf{j} \rangle = \hat{\sigma}_e \langle \mathbf{e} \rangle, \tag{1.2}$$

as well as the statistical moments of the electric fields.

We represent the random functions in the form of the sum of the mathematical expectations and the fluctuations

$$\mu = \langle \mu \rangle + \mu_0 \; (\mu = p, q, \mathbf{e}, \mathbf{j}, \ldots)$$

and introduce the following notation for the two-point moments of the field $\{\mathbf{e}, p, q\}$:

$$\begin{aligned} \langle p_0^m (M) q_0^n (M) \mathbf{e}_0 (N) p_0 (N) \rangle &= \mathbf{A}^{m+1,n} (\mathbf{r}), \\ \langle p_0^m (M) q_0^n (M) \mathbf{e}_0 (N) q_0 (N) \rangle &= \mathbf{A}^{m,n+1} (\mathbf{r}), \\ \langle p_0^m (M) q_0^n (M) p_0 (N) \rangle &= D^{m+1,n} (\mathbf{r}), \\ \langle p_0^m (M) q_0^n (M) q_0 (N) \rangle &= D^{m,n+1} (\mathbf{r}). \end{aligned} \tag{1.3}$$

In this notation, the averaged Ohm's law (1.2) can be represented in the form

$$\langle j_i \rangle = (\langle p \rangle \delta_{ik} + \langle q \rangle \varepsilon_{ik}) \langle e_k \rangle + \delta_{ik} A_k^{1,0} (0) + \varepsilon_{ik} A_k^{0,1} (0), \tag{1.4}$$

where δ_{ik} is a unit tensor; ε_{ik} is a unit antisymmetric tensor. From this it can be seen that, to find the effective parameters, it is necessary to determine single-point moments of the second order $\mathbf{A}^{1,0}(0)$, $\mathbf{A}^{0,1}(0)$.

From Eqs. (1.1) integral recurrence equations can be obtained, connecting the single-point moments with two-point moments of higher order,

$$A_i^{m,n} (0) = -\int\limits_S \frac{\partial G}{\partial x_i} \frac{\partial}{\partial x_l} \{ [D^{m+1,n}(\mathbf{r}) \delta_{lk} + D^{m,n+1}(\mathbf{r}) \varepsilon_{lk}] \langle e_k \rangle + \delta_{lk} A_k^{m+1,n} (\mathbf{r}) + \varepsilon_{lk} A_k^{m,n+1} (\mathbf{r}) \} \, dS; \tag{1.5}$$

G is Green's function.

Equations (1.3) were obtained using an integral representation of the solution of Eqs. (1.1), multiplying them by $p_0^m q_0^n$ with subsequent averaging. These recurrence equations are completely solvable if, as is assumed in the theory of turbulence, we use representations of the moments in terms of the principal invariants of the vector fields.

The field $\{\mathbf{e}, p, q\}$ has axial symmetry in the sense that there is a definite direction of the mean field $\langle \mathbf{e} \rangle$; therefore, the two-point moments of the function of the distribution of this field can be represented in the form

$$A_i^{m,n} (\mathbf{r}) = [(a_1^{m,n} \delta_{ik} + a_2^{m,n} \varepsilon_{ik}) \lambda_k + (a_3^{m,n} \delta_{ik} + a_4^{m,n} \varepsilon_{ik}) \gamma_k] \, |\langle \mathbf{e} \rangle|. \tag{1.6}$$

In expression (1.6), λ is a unit vector in the direction of the mean field $\langle \mathbf{e} \rangle$; γ is a unit vector in a direction between the two points, for which the moments are considered; $a_\alpha^{m,n}$ ($\alpha = 1, 2, 3, 4$) are scalar functions of the following three arguments: r, $\gamma_i \lambda_k \delta_{ik}$, $\gamma_i \lambda_k \varepsilon_{ik}$. Here it is taken into consideration that, for a random vector $\mathbf{e}$ in a magnetic field, the condition of invariance with respect to transformation of the mirror image will not be satisfied. This is connected with the fact that the tensor of the conductivity $\hat{\sigma}$ in Ohm's law (1.1) contains antisymmetrical terms, which change sign with transformation of the mirror image. Therefore, using this transformation, it would be necessary to simultaneously change the direction of the magnetic field to the opposite.

For axisymmetrical random fields, the values of the moments at zero and at infinity should not depend on the direction between the two points. Therefore, we can write

$$a_3^{m,n} = a_4^{m,n} = 0 \text{ with } r = 0, \infty.$$

From this we obtain the result that the components of the single-point moments $a_1^{m,n}(0)$ and $a_2^{m,n}(0)$ represent, respectively, the longitudinal and transverse components of the vector $\mathbf{A}^{m,n}(0)$ to the vector $\langle \mathbf{e} \rangle$.

Substituting (1.6) into Eqs. (1.5), the system of integral recurrence equations can be brought to a system of algebraic recurrence equations

$$D^{m+1,n}(0) + a_1^{m+1,n}(0) - a_2^{m,n+1}(0) - D^{m,n}(0)\left[a_1^{1,0}(0) - a_2^{0,1}(0)\right] + 2\langle p\rangle a_1^{m,n}(0) - R_1^{m+1,n} - R_2^{m,n+1} = 0,$$

$$D^{m,n+1}(0) + a_1^{m,n+1}(0) + a_2^{m+1,n}(0) - D^{m,n}(0)\left[a_1^{0,1}(0) + a_2^{1,0}(0)\right] + 2\langle p\rangle a_2^{m,n}(0) + R_1^{m,n+1} - R_2^{m+1,n} = 0, \quad (1.7)$$

where $R_{1,2}^{m,n}$ are definite integrals of the values of the coefficients of the expansion of the functions $a_\alpha^{m,n}(\mathbf{r})$ in Fourier series in terms of the angle between the vectors λ and γ.

Equations (1.7) and the representations of the moments (1.6) are valid for an arbitrary function of the distribution of the parameters of an inhomogeneous medium p and q. In the case of a two-phase medium with the relative volumetric concentrations of the phases c_1 and c_2 ($c_1 + c_2 = 1$) and with β = const, Eqs. (1.7) are simplified. Neglecting the terms $R_2^{m,n}$ for m = 1, n = 0, and m = 0, n = 1, and using the expression of the averaged Ohm's law (1.4), we obtain a relationship connecting the effective parameters p_0 and q_0.

$$2\beta(1+\beta^2)(c_2-c_1)p_e\Delta + 2(1+\beta^2)q_e = \beta(1-\Delta^2)(\sigma_1+\sigma_2)$$
$$(\Delta = (\sigma_1-\sigma_2)/(\sigma_1+\sigma_2)).$$

Together with the supplementary expression [5]

$$4\beta(1+\beta^2)(p_e^2+q_e^2) - 4(1+\beta^2)(\sigma_1+\sigma_2)q_e + \beta(1-\Delta^2)(\sigma_1+\sigma_2)^2 = 0$$

we obtain a system of two algebraic equations for determining the effective parameters with arbitrary concentrations of the phases and fluctuations of the conductivity. These expressions correspond completely to the solution obtained in [10] by the method of self-consistency of the local fields.

Without dwelling on an analysis of the effective parameters, let us examine in more detail the question of the properties of the mean values of the electric fields of a randomly inhomogeneous medium in strong magnetic fields.

For determination of the mean fields in the phases $\langle \mathbf{e}\rangle_1$ and $\langle \mathbf{e}\rangle_2$, it is necessary to calculate the single-point moment of the second order $\mathbf{A}^{1,0}(0)$. In accordance with (1.3), we can write

$$\mathbf{A}^{1,0}(0) = \int\int (p - \langle p\rangle)(\langle \mathbf{e}\,|\,p, q\rangle - \langle \mathbf{e}\rangle) f(p, q)\,dp\,dq, \quad (1.8)$$

where $\langle \mathbf{e}|p, q\rangle$ is the relative mathematical expectation of $\mathbf{e}$; f(p, q) is the density of the distribution of the parameters p and q.

In the adopted model of a two-phase medium, using a δ function, $f(p \cdot q)$ can be represented in the form

$$f(p, q) = c_1\delta(p-p_1)\delta(q-q_1) + c_2\delta(p-p_2)\delta(q-q_2). \quad (1.9)$$

From (1.8) and (1.9), we find

$$\mathbf{A}^{1,0}(0) = c_1c_2(p_1-p_2)(\langle \mathbf{e}\rangle_1 - \langle \mathbf{e}\rangle_2).$$

From this, taking into consideration that

$$\langle \mathbf{e}\rangle = c_1\langle \mathbf{e}\rangle_1 + c_2\langle \mathbf{e}\rangle_2,$$

we have

$$\langle \mathbf{e}\rangle_1 = \langle \mathbf{e}\rangle + \frac{\mathbf{A}^{1,0}(0)}{c_1(p_1-p_2)}, \quad \langle \mathbf{e}\rangle_2 = \langle \mathbf{e}\rangle - \frac{\mathbf{A}^{1,0}(0)}{c_2(p_1-p_2)}.$$

Using formulas (1.4) and (1.6) with $\sigma_1 \neq \sigma_2$ and β = const, we finally obtain

$$\langle e_i\rangle_1 = \left\{\left[1 + \frac{p_e - \langle p\rangle + \beta(q_e - \langle q\rangle)}{c_1(\sigma_1 - \sigma_2)}\right]\delta_{ik} + \frac{q_e - \langle q\rangle - \beta(p_e - \langle p\rangle)}{c_1(\sigma_1 - \sigma_2)}\varepsilon_{ik}\right\}\langle e_k\rangle,$$

$$\langle e_i\rangle_2 = \left\{\left[1 - \frac{p_e - \langle p\rangle + \beta(q_e - \langle q\rangle)}{c_2(\sigma_1 - \sigma_2)}\right]\delta_{ik} - \frac{q_e - \langle q\rangle - \beta(p_e - \langle p\rangle)}{c_2(\sigma_1 - \sigma_2)}\varepsilon_{ik}\right\}\langle e_k\rangle.$$

As can be seen from these formulas, to calculate the mean fields $\langle e\rangle_1$ and $\langle e\rangle_2$, it is necessary to know only the effective parameters p_e and q_e.

Analogous expressions can be obtained for the mean values of the current density $\langle j\rangle_1$ and $\langle j\rangle_2$.

To show the effect of the magnetic field and the concentration of inhomogeneities on the distribution of the electric fields, Fig. 1 gives dependences of the modulus of the mean values of the field in one phase and dependences on the concentration of inhomogeneities with a relatively small fluctuation of the conductivity $\Delta = 0.3$. As can be seen from the curves, if the concentration of the phase is less than half of the whole composition ($c_1 < 0.5$), the value of the mean concentration of the field in the phase in this case decreases and, correspondingly, increases with $c_1 > 0.5$. This means that the electric field is, so to speak, "displaced" from the phase whose volume is less to the other phase of greater volume. The sharpest change in $|\langle e\rangle_1|$ takes place near $c_1 = 0.5$. At the limit of strong magnetic fields ($\beta \to \infty$), the dependence changes jumpwise from zero to $2|\langle e\rangle|$ with $c_1 = c_2 = 0.5$ (heavy line).

The effect of fluctuation of the conductivity with different magnetic fields is reflected in Figs. 2 and 3. In the phase with the lower concentration ($c_1 = 0.35$) in the absence of a magnetic field $|\langle e\rangle_1|$ changes smoothly from $2|\langle e\rangle|$, where the conductivity $\sigma_1 \to \infty$. In a strong magnetic field the character of the dependence changes sharply. The mean intensity of the electric field in the first phase tends toward zero with any given fluctuations of the conductivity, with the exception of those cases where $\Delta \approx 0$. All the values of $|\langle e\rangle_1|$ lie between the curves for $\beta = 0$ and $\beta \to \infty$ (heavy lines). Correspondingly, in the second phase (Fig. 3), with any given fluctuations of the conductivity, the mean intensity of the electric field rises with an increase in β. All the values of $|\langle e\rangle_2|$ lie between the curves for $\beta = 0$ and $\beta \to \infty$.

2. As follows from an analysis near the flow-through threshold ($c_1 = c_2 = 0.5$) there are large fluctuations of the electric field, which rise considerably in strong magnetic fields. To explain this state of an inhomogeneous system, let us examine its characteristics in more detail. To this end, let us consider an inhomogeneous medium with a dual-periodic distribution of the inhomogeneities with identical periods along two axes, which allows an analysis also of the local structure of the fields.

In the case where, at the boundaries of cells with different conductivities, the conditions of ohmic contact are satisfied, from the solution of the boundary-value problem for Eqs. (1.1) we obtain expressions for the current density in two adjacent square cells ($0 < x < l$, $0 < y < l$), and ($0 < x < l$, $-l < y < 0$)

$$j_1(z) = j_{x_1} - ij_{y_1} = |B|\left[|C_1|(-1)^{\frac{1}{2}\left(\frac{1}{2}-\gamma-\alpha+\delta\right)}X + |C_2|(-1)^{\frac{1}{2}\left(-\frac{1}{2}+\gamma-\alpha+\delta\right)}X^{-1}\right] \quad (z = x + iy), \tag{2.1}$$

$$j_2 = \sqrt{\frac{\sigma_2}{\sigma_1}}|B|\left[|C_1|(-1)^{\frac{1}{2}\left(\frac{1}{2}+\gamma+\alpha+\delta\right)}X + |C_2|(-1)^{\frac{1}{2}\left(-\frac{1}{2}-\gamma+\alpha+\delta\right)}X^{-1}\right],$$

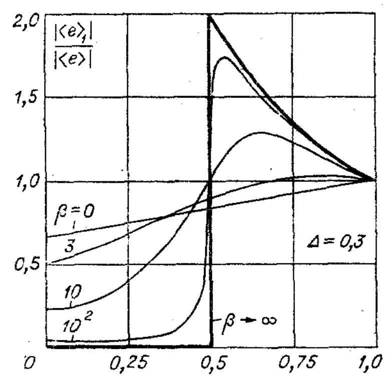

Fig. 1

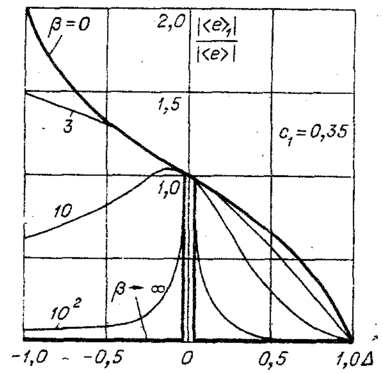

Fig. 2

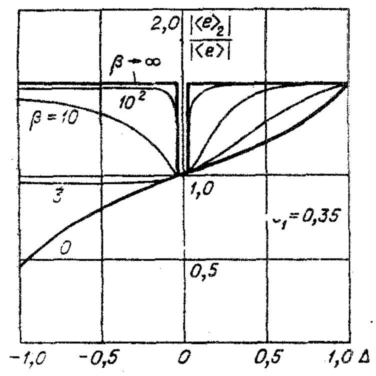

Fig. 3

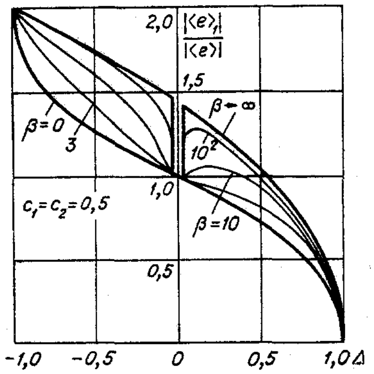

Fig. 4

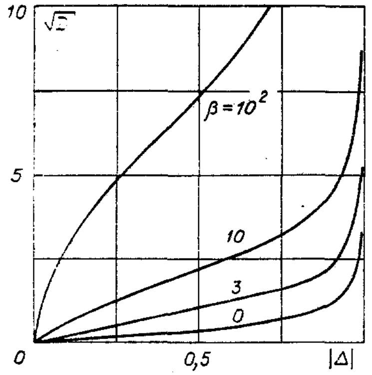

Fig. 5

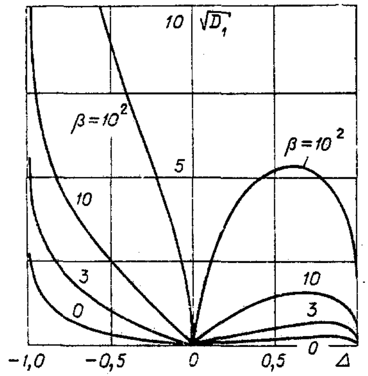

Fig. 6

where

$$|B| = \frac{1}{2\sigma_1}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_1\beta_2-\sigma_2\beta_1)^2}; \qquad \gamma = \frac{1}{\pi}\,\mathrm{arctg}\,|B|\sqrt{\frac{\sigma_1}{\sigma_2}};$$

$$\alpha = \frac{1}{\pi}\,\mathrm{arctg}\frac{\sigma_2\beta_1-\sigma_1\beta_2}{\sigma_1+\sigma_2}; \qquad \delta = \frac{1}{\pi}\,\mathrm{arctg}\frac{\sigma_2\beta_1-\sigma_1\beta_2}{\sigma_1-\sigma_2};$$

$$X = \left[\frac{\mathrm{cn}\,u}{\mathrm{sn}\,u\cdot\mathrm{dn}\,u}\right]^{2\gamma},$$

where u is the argument of the elliptical Jacobi functions sn u, cn u, and dn; K is an analytical integral of the first kind; k is the modulus of the elliptical integral; for the case of square cells under consideration here, we have K = 1.8541, k = $\sqrt{0.5}$. The real constants $|C_1|$ and $|C_2|$ are determined by assignment of the electrical current J, which flows through the whole system.

As follows from (2.1), with $|C_2| = 0$, the local values in two adjacent cells are connected by relationships of symmetry

$$\bar{j}_2(\bar{z}) = \sqrt{\frac{\sigma_2}{\sigma_1}}(-1)^{-\frac{1}{2}-\delta} j_1(z).$$

Relationships of symmetry of this type are also valid for values averaged over the cells:

$$\langle j\rangle_2 = \sqrt{\frac{\sigma_2}{\sigma_1}}(-1)^{\alpha}\langle j\rangle_1.$$

Using the existing symmetry, we can determine the mean values of the electric field in the cells and the effective parameters of the medium as a whole. For one of the phases we obtain formulas, which connect the mean values in the phase with the mean field over the whole system

$$\langle j\rangle_1 = 2\left(1+\sqrt{\frac{\sigma_2}{\sigma_1}}e^{i\pi\alpha}\right)\langle j\rangle, \quad \langle e\rangle_1 = 2\sqrt{\frac{\sigma_2}{\sigma_1}}\left(\sqrt{\frac{\sigma_2}{\sigma_1}}+e^{i\pi\alpha}\right)\langle e\rangle.$$

Analogous expressions can be obtained for the second phase.

From these formulas it can be seen that, if the conductivity of any given phase decreases, the current is displaced from it, and the intensity of the field rises correspondingly, not exceeding, however, the doubled value of the mean field. It is found that the Joule dissipation is identical with any given magnetic field and fluctuations of the conductivity.

In weak magnetic fields, even small differences in the conductivities of the phases lead to considerable changes in the mean intensity of the electrical field (Fig. 4).

The value of the fluctuations of the fields can be judged from an examination of the dispersion of the field in individual cells and in the system as a whole. With β = const, the dispersion of the field in the whole system is determined by the expression

$$D = \frac{\langle |e|^2 \rangle - |\langle e \rangle|^2}{|\langle e \rangle|^2} = \frac{\sqrt{1+\beta^2\Delta^2} - \sqrt{1-\Delta^2}}{\sqrt{1-\Delta^2}}. \qquad (2.2)$$

For individual phases, the dispersion is given by the formula

$$D_1 = \frac{\langle |e|^2 \rangle_1 - |\langle e \rangle_1|^2}{|\langle e \rangle|^2} = \sqrt{1+\beta^2\Delta^2} \left[\sqrt{\frac{1-\Delta}{1+\Delta}} - \frac{2(1-\Delta)}{\sqrt{1+\beta^2\Delta^2} + \sqrt{1-\Delta^2}} \right]. \qquad (2.3)$$

In strong magnetic fields ($\beta \to \infty$), D_1 and D_2 vary proportionally to the parameter $\beta\Delta$, which is not small with relatively small fluctuations of the conductivity Δ. The rise in the dispersion in limiting cases $|\Delta| \to 1$ takes place due to only one phase, while, at the same time, for the second phase it is finite, as can be seen from Figs. 5 and 6, plotted using formulas (2.2) and (2.3).

From the curves given the conclusion can be drawn that there is a gradientless rise in the mean-square fluctuations of the electric field near the flow-through threshold ($|\Delta| \to 1$). In this case the values of the mean fields are found to be finite (see Fig. 4).

Thus, with a study of the statistical properties of fields near the flow-through threshold, it is insufficient to know only moments of the first order; far more information on the special characteristics of the distribution of the field is given by moments of the second order.

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